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TOPICAL ISSUE

Nonfragile Filtering under Bounded Exogenous Disturbances

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Abstract—This paper considers filtering for linear systems subjected to persistent exogenous disturbances. The filtering quality is characterized by the size of the bounding ellipsoid that contains the estimated output of the system. A regular approach is proposed to solve the nonfragile filtering problem. This problem consists in designing a filter matrix that withstands admissible variations of its coefficients. The concept of invariant ellipsoids is applied to reformulate the original problem in terms of linear matrix inequalities and reduce it to a parametric semidefinite programming problem easily solved numerically. This paper continues the series of author's research works devoted to filtering under nonrandom bounded exogenous disturbances and measurement errors.

Keywords: linear control system, exogenous disturbances, filtering, nonfragility, Luenberger observer, linear matrix inequalities (LMIs), invariant ellipsoids

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1. INTRODUCTION

The filtering problem consists in estimating the state of a dynamic system from available measurements. Under random disturbances, it admits an almost exhaustive solution using the Kalman filter. However, in many situations, the randomness assumption becomes unreasonable for noises. Often, one knows only that the disturbances are bounded but otherwise arbitrary; in this case, the *guaranteed* estimates of the states can be constructed. This approach goes back to the research works of Witzenhausen, Bertsekas and Rhodes, and Schweppe [1]. The ellipsoidal filtering technique was developed by Russian scholars [2, 3].

Later, based on the technique of linear matrix inequalities (LMIs) [5, 6] and the concept of invariant ellipsoids, the author jointly with Polyak studied the filtering problem for stationary problems with bounded nonrandom disturbances [4]. In the class of linear time-invariant filters, the problem turned out to be completely solvable: an optimal filter was designed, and a uniform state estimation was obtained: its error is surely contained in a single ellipsoid for all time instants. This topic was further developed in [7–9].

On the other hand, uncertainty is inevitably introduced into a control system due to the technical implementation imperfections of the controller or the need to tune its parameters during operation. Even small perturbations of the optimal controller's gains may violate its stabilizability property [10]. This phenomenon was called *fragility*, and it was subsequently studied in various problem statements; for example, see [11]. An approach to design the so-called *nonfragile controllers* (the ones withstanding variations in their parameters) was proposed in [12, 13], as applied to the suppression of nonrandom bounded disturbances.

This paper continues both lines of research. We describe a regular approach to *nonfragile filtering*, i.e., designing a filter matrix that withstands *admissible* variations in its coefficients. As it turns out, even small perturbations of the optimal filter matrix may violate the invariance of the ellipsoid (obtained under the assumption of its exact implementation) containing the system residual: the residual trajectories may leave this ellipsoid. The aim of this paper is to construct the so-called *nonfragile pair*, i.e., the filter matrix and the corresponding ellipsoid (as small as possible) that contains the system residual under all admissible perturbations of the filter matrix.

2. THE FILTERING PROBLEM: STATEMENT AND SOLUTION

Let us recall the statement and solution of the filtering problem with bounded exogenous disturbances. Consider a linear continuous-time dynamic system described by

$$\begin{aligned} \dot{x} &= Ax + D_1 w, \quad x(0) = x_0, \\ y &= Cx + D_2 w, \\ z &= C_1 x, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$, $C_1 \in \mathbb{R}^{r\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^l$ is the observed output, $z(t) \in \mathbb{R}^r$ is the estimated output, and $w(t) \in \mathbb{R}^m$ is an exogenous disturbance (noise) satisfying the constraint[1]

$$\|w(t)\| \leqslant 1 \quad \text{for all } t \geqslant 0.$$

Although the disturbances in the system state and output generally have different nature, it is convenient to consider them the same, supposing that the matrices D_1 and D_2 "cut out" different pieces from the vector w. The pair (A, C) is assumed to be observable.

Let the system state x be unmeasurable, and let the information about the system be provided by its output y. We design a filter described by a linear differential equation with respect to the state estimate $\widehat{x}$ that includes the discrepancy between the output y and its forecast $C\widehat{x}$:

$$\dot{\widehat{x}} = A\widehat{x} + L(y - C\widehat{x}), \quad L \in \mathbb{R}^{n\times l}. \tag{2}$$

The filter (2) has the same structure as the well-known Luenberger observer [14, 15]: it is linear and time-invariant, and only the constant matrix L (the so-called *filter matrix*) has to be chosen here.

We introduce the *residual* $e(t) = x(t) - \widehat{x}(t)$; according to (1) and (2), it satisfies the differential equation[2]

$$\dot{e} = (A - LC)e + (D_1 - LD_2)w. \tag{3}$$

Then the accuracy of filtering (estimating the output z) will be characterized by the value

$$e_1 = z - \widehat{z} = C_1(x - \widehat{x}) = C_1 e.$$

The problem is to find the minimal (in a certain sense) single ellipsoid containing the residual e_1.

The apparatus of LMIs and the ideology of invariant ellipsoids [5, 6] is a convenient technical tool for solving this problem. Recall that an ellipsoid centered at the origin

$$\mathcal{E} = \Big\{ x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \leqslant 1 \Big\}, \quad P \succ 0$$

[1] Throughout this paper, $\|\cdot\|$ indicates the Euclidean vector norm and, simultaneously, the spectral matrix norm, $\mathbb{S}^n$ is the space of symmetric matrices of order n, and I denotes an identity matrix of appropriate dimensions. All matrix inequalities are understood in the sign definiteness sense.

[2] Note that the filter matrix L stabilizes system (3). The existence of a matrix L making the matrix $(A - LC)$ stable (Hurwitz) follows from the observability of the original system (1).

is said to be *invariant* for a dynamic system $\dot{x} = Ax + Dw$ if the condition $x(0) \in \mathcal{E}$ implies $x(t) \in \mathcal{E}$ for all time instants $t \geqslant 0$. Moreover, given an invariant ellipsoid $\mathcal{E}$ with a matrix P, the linear output $z(t) = Cx(t) \in \mathbb{R}^r$ of this dynamic system will belong to the *bounding* ellipsoid

$$\mathcal{E}_z = \Big\{ z \in \mathbb{R}^r : \quad z^{\mathrm{T}} \big(CPC^{\mathrm{T}}\big)^{-1} z \leqslant 1 \Big\}$$

if $x(0) \in \mathcal{E}$ and tend to it if $x(0) \notin \mathcal{E}$.

Thus, the ideology of invariant and bounding ellipsoids allows estimating the t-uniform accuracy of filtering under small deviations and the asymptotic accuracy of filtering under large deviations. Within this approach, for a fixed matrix L, we find the minimal bounding ellipsoid and then minimize it with respect to L. A conventional optimality criterion for bounding ellipsoids is the *trace criterion* $f(P) = \operatorname{tr} CPC^{\mathrm{T}}$, which corresponds to the sum of the squares of its semi-axes.

The following result was established in [4].

Theorem 1. *Let Q_* and Y_* be the solution of the optimization problem*

$$\min \operatorname{tr} H$$

subject to the constraints

$$\begin{pmatrix} A^{\mathrm{T}}Q + QA - YC - C^{\mathrm{T}}Y^{\mathrm{T}} + \alpha Q & QD_1 - YD_2 \\ D_1^{\mathrm{T}}Q - D_2^{\mathrm{T}}Y^{\mathrm{T}} & -\alpha I \end{pmatrix} \preccurlyeq 0,$$

$$\begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0, \qquad Q \succ 0$$

with respect to the matrix variables $Q \in \mathbb{S}^n$, $Y \in \mathbb{R}^{n \times l}$, and $H \in \mathbb{S}^r$ and the scalar parameter $\alpha > 0$.

Then the optimal filter matrix is given by

$$L_* = Q_*^{-1} Y_*,$$

and the minimal bounding ellipsoid containing the estimation error of the output z of system (1) *with $x_0 = 0$ is defined by the matrix*

$$C_1 Q_*^{-1} C_1^{\mathrm{T}}.$$

Note that for a fixed α, this problem reduces to a semidefinite programming problem easily solved numerically.

Remark 1. If the initial state $x(0) = x_0$ of system (1) is known, then the optimization problem of Theorem 1 includes the additional constraint

$$x_0^{\mathrm{T}} Q x_0 \leqslant 1,$$

meaning that $e(0) = x(0) - \widehat{x}(0) = x_0 \in \mathcal{E}$.

If the initial point belongs to some *initial state ellipsoid*

$$x(0) \in \mathcal{E}_0 = \Big\{ x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P_0^{-1} x \leqslant 1 \Big\}, \quad P_0 \succ 0,$$

then the additional constraint is the LMI

$$Q \preccurlyeq P_0^{-1},$$

meaning that $\mathcal{E}_0 \subset \mathcal{E}$. In this case, we again have $e(0) = x(0) - \widehat{x}(0) = x(0) \in \mathcal{E}_0 \subset \mathcal{E}$.

Thus, in both cases, the accuracy of filtering is estimated uniformly.

3. INVARIANCE AND NONFRAGILITY

We say that a filter matrix L and a positive definite matrix $P = Q^{-1}$ form a *nonfragile pair* with a nonfragility level γ if, for any $\Delta\colon \|\Delta\| \leqslant \gamma$, the disturbed filter matrix $(L+\Delta)$ stabilizes system (3) and the matrix P defines its invariant ellipsoid. In this case, the filter and the corresponding bounding ellipsoid containing the estimation error of the system output will be both called *nonfragile*. As before, we will strive to make this ellipsoid as small as possible.

Let us formulate the main result of the paper.

Theorem 2. *Let $\widetilde{Q}$ and $\widetilde{Y}$ be the solution of the optimization problem*

$$\min \operatorname{tr} H$$

subject to the constraints

$$\begin{pmatrix} A^{\mathrm{T}}Q + QA - YC - C^{\mathrm{T}}Y^{\mathrm{T}} + \alpha Q + \varepsilon C^{\mathrm{T}}C & QD_1 - YD_2 + \varepsilon C^{\mathrm{T}}D_2 & \gamma Q \\ D_1^{\mathrm{T}}Q - D_2^{\mathrm{T}}Y^{\mathrm{T}} + \varepsilon D_2^{\mathrm{T}}C & -\alpha I + \varepsilon D_2^{\mathrm{T}}D_2 & 0 \\ \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0,$$

$$\begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0, \qquad Q \succ 0$$

with respect to the matrix variables $Q \in \mathbb{S}^n$, $Y \in \mathbb{R}^{n\times l}$, and $H \in \mathbb{S}^r$, the scalar variable ε, and the scalar parameter $\alpha > 0$.

Then the matrix

$$C_1 \widetilde{Q}^{-1} C_1^{\mathrm{T}}$$

defines a nonfragile bounding ellipsoid for the estimation error of the output z of system (1) *with $x_0 = 0$ that corresponds to the nonfragile pair*

$$\widetilde{L} = \widetilde{Q}^{-1}\widetilde{Y}, \qquad \widetilde{P} = \widetilde{Q}^{-1}$$

with the nonfragility level γ.

The proofs of Theorems 2 and 4 (see below) are provided in the Appendix.

As before, the problem formulated in Theorem 2 is a parametric semidefinite programming problem easily solved numerically.

The approach below is based on constructing a common quadratic Lyapunov function for the uncertain system and gives only sufficient conditions for robust asymptotic stability. We will not analyze in detail the degree of conservatism of the resulting ellipsoidal estimate. However, numerical examples indicate that conservatism is not very great.

Remark 2. Note that the nonfragile filter matrix L robustly stabilizes the system

$$\dot{e} = (A - (L+\Delta)C)e + (D_1 - (L+\Delta)D_2)w$$

under all admissible uncertainties $\Delta\colon \|\Delta\| \leqslant \gamma$. With such a special structure of the closed-loop system matrix, robust stabilization is possible for any γ (for example, see [6, Remark 5.2.1]). In other words, the nonfragility level γ can be set arbitrarily large, which will only increase the size of the nonfragile invariant ellipsoid.

4. THE DISCRETE-TIME CASE

Consider the linear discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + D_1 w_k, \\ y_k &= Cx_k + D_2 w_k, \\ z_k &= C_1 x_k \end{aligned} \tag{4}$$

with the initial condition x_0, where $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{l\times n}$, $C_1 \in \mathbb{R}^{r\times n}$, $D_1 \in \mathbb{R}^{n\times m}$, $D_2 \in \mathbb{R}^{l\times m}$, $x_k \in \mathbb{R}^n$ is the state, $y_k \in \mathbb{R}^l$ is the observed output, $z_k \in \mathbb{R}^r$ is the estimated output, and $w_k \in \mathbb{R}^m$ is an exogenous disturbance satisfying the constraint

$$\|w_k\| \leqslant 1 \quad \text{for all } k = 0, 1, 2, \ldots.$$

By assumption, the pair (A, C) is observable.

As in the continuous-time case, we design a filter described by a linear difference equation with respect to the state estimate $\widehat{x}_k$ that includes the discrepancy between the output y and its forecast $C\widehat{x}_k$:

$$\widehat{x}_{k+1} = A\widehat{x}_k + L(y_k - C\widehat{x}_k), \quad L \in \mathbb{R}^{n\times l}.$$

Here, a constant filter matrix L has to be chosen.

The definition of the minimal bounding ellipsoid containing the estimation error

$$e_{1,k} = z_k - \widehat{z}_k = C_1 e_k,$$

where $e_k = x_k - \widehat{x}_k$ is the system residual, remains essentially the same as in the continuous-time case. This ellipsoid can be found using the following result, representing a discrete-time analog of Theorem 1.

Theorem 3 [4, 6]. *Let Q_* and Y_* be the solution of the optimization problem*

$$\min \operatorname{tr} H$$

subject to the constraints

$$\begin{pmatrix} -\alpha Q & (QA - YC)^{\mathrm{T}} & 0 \\ QA - YC & -Q & QD_1 - YD_2 \\ 0 & (QD_1 - YD_2)^{\mathrm{T}} & -(1-\alpha)I \end{pmatrix} \preccurlyeq 0, \quad \begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0, \quad Q \succ 0$$

with respect to the matrix variables $Q \in \mathbb{S}^n$, $Y \in \mathbb{R}^{n\times l}$, and $H \in \mathbb{S}^r$ and the scalar parameter $0 < \alpha < 1$.

Then the optimal filter matrix is given by

$$L_* = Q_*^{-1} Y_*,$$

and the minimal bounding ellipsoid containing the estimation error of the output z of system (4) with $x_0 = 0$ is defined by the matrix

$$C_1 Q_*^{-1} C_1^{\mathrm{T}}.$$

A nonfragile pair is defined by analogy with the continuous-time case; it can be found using the following result.

Theorem 4. *Let $\widetilde{Q}$ and $\widetilde{Y}$ be the solution of the optimization problem*

$$\min \operatorname{tr} H$$

subject to the constraints

$$\begin{pmatrix} -\alpha Q + \varepsilon C^{\mathrm{T}} C & (QA - YC)^{\mathrm{T}} & \varepsilon C^{\mathrm{T}} D_2 & 0 \\ QA - YC & -Q & QD_1 - YD_2 & \gamma Q \\ \varepsilon D_2^{\mathrm{T}} C & (QD_1 - YD_2)^{\mathrm{T}} & -(1-\alpha) I + \varepsilon D_2^{\mathrm{T}} D_2 & 0 \\ 0 & \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0,$$

$$\begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0, \quad Q \succ 0$$

with respect to the matrix variables $Q \in \mathbb{S}^n$, $Y \in \mathbb{R}^{n\times l}$, and $H \in \mathbb{S}^r$, the scalar variable ε, and the scalar parameter $0 < \alpha < 1$.

Then the matrix

$$C_1 \widetilde{Q}^{-1} C_1^{\mathrm{T}}$$

defines a nonfragile bounding ellipsoid for the estimation error of the output z_k of system (4) with $x_0 = 0$ that corresponds to the nonfragile pair

$$\widetilde{L} = \widetilde{Q}^{-1}\widetilde{Y}, \qquad \widetilde{P} = \widetilde{Q}^{-1}$$

with the nonfragility level γ.

As in the continuous-time case, the optimization problem of Theorem 4 is a simple parametric semidefinite programming problem.

Also, we emphasize the validity of Remark 2 in the discrete-time case. That is, the system

$$e_{k+1} = (A - (L+\Delta)C)e_k + (D_1 - (L+\Delta)D_2)w_k$$

can be robustly stabilized under all admissible uncertainties $\Delta\colon \|\Delta\| \leqslant \gamma$ for any nonfragility level γ.

5. EXAMPLE

We demonstrate the invariant ellipsoids-based filtering approach for bounded exogenous disturbances on an example of estimating the state of a double-spring pendulum (Fig. 1).

Let x_1 and x_2 denote the coordinates of the left and right body, respectively, and let v_1 and v_2 be their velocities. The right body is affected by an exogenous disturbance w_1. The disturbed

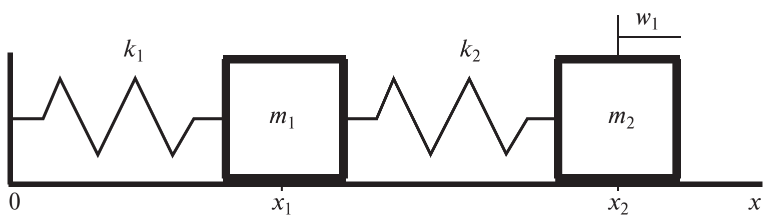

Fig. 1. A double-spring pendulum.

oscillations of the system are described by the continuous-time model

$$\begin{aligned}\dot{x}_1 &= v_1,\\ \dot{x}_2 &= v_2,\\ \dot{v}_1 &= -\frac{k_1+k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2,\\ \dot{v}_2 &= \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_2 + \frac{1}{m_2}w_1,\end{aligned}$$

where k_1 and k_2 indicate the stiffness coefficients of the left and right spring, respectively, and m_1 and m_2 are the masses of the left and right body, respectively.

Selecting the state vector

$$x = \begin{pmatrix} x_1\\ x_2\\ v_1\\ v_2\end{pmatrix},$$

the observed output

$$y = \begin{pmatrix} x_1\\ x_2\end{pmatrix} + 0.1\begin{pmatrix} w_2\\ w_3\end{pmatrix}$$

(the noisy coordinates), and the estimated output

$$z = \begin{pmatrix} v_1\\ v_2\end{pmatrix}$$

(the velocities), we arrive at system (1) with the matrices

$$A = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\dfrac{k_1+k_2}{m_1} & \dfrac{k_2}{m_1} & 0 & 0\\ \dfrac{k_2}{m_2} & -\dfrac{k_2}{m_2} & 0 & 0\end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ \dfrac{1}{m_2} & 0 & 0\end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0.1 & 0\\ 0 & 0 & 0.1\end{pmatrix}, \qquad C_1 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

In addition,

$$w = \begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix}, \qquad \|w\| \leqslant 1.$$

For the sake of simplicity, we assume the unity system parameters $k_1 = k_2 = m_1 = m_2 = 1$ and take $P_0 = 0.15I$.

First, using Theorem 1, we find the solution without the nonfragility requirement. The result is the filter matrix

$$L_* = \begin{pmatrix} 1.4808 & 0.2309\\ -0.1641 & 2.1590\\ -0.5457 & 1.0867\\ 0.6232 & 3.4354\end{pmatrix}$$

and the matrix

$$Q_* = \begin{pmatrix} 5.0166 & -0.1455 & -2.0847 & -0.0184 \\ -0.1455 & 6.1854 & -0.1544 & -1.5832 \\ -2.0847 & -0.1544 & 4.0310 & 0.0762 \\ -0.0184 & -1.5832 & 0.0762 & 1.3265 \end{pmatrix};$$

the corresponding minimal bounding ellipse $\mathcal{E}_*$ with the matrix

$$C_1 Q_*^{-1} C_1^{\mathrm{T}} = \begin{pmatrix} 0.3167 & -0.0046 \\ -0.0046 & 1.0863 \end{pmatrix}$$

contains the estimation error e_1 of the output z.

Next, we set the nonfragility level

$$\gamma = 2.$$

Solving the optimization problem of Theorem 2 yields the filter matrix

$$\widetilde{L} = \begin{pmatrix} 23.3910 & 0.9878 \\ 0.9883 & 21.9974 \\ 14.5498 & 1.0207 \\ 0.9240 & 26.6793 \end{pmatrix}$$

and the matrix

$$\widetilde{Q} = \begin{pmatrix} 5.3807 & -0.1112 & -2.0697 & -0.0284 \\ -0.1112 & 6.2215 & -0.1462 & -1.5363 \\ -2.0697 & -0.1462 & 3.3329 & 0.0697 \\ -0.0284 & -1.5363 & 0.0697 & 1.2650 \end{pmatrix}.$$

The resulting nonfragile pair is $(\widetilde{L}, \widetilde{P})$, where

$$\widetilde{P} = \begin{pmatrix} 0.2446 & 0.0103 & 0.1522 & 0.0097 \\ 0.0103 & 0.2301 & 0.0107 & 0.2790 \\ 0.1522 & 0.0107 & 0.3951 & -0.0054 \\ 0.0097 & 0.2790 & -0.0054 & 1.1299 \end{pmatrix};$$

the corresponding nonfragile bounding ellipse $\widetilde{\mathcal{E}}$ with the matrix

$$C_1 \widetilde{Q}^{-1} C_1^{\mathrm{T}} = \begin{pmatrix} 0.3951 & -0.0054 \\ -0.0054 & 1.1299 \end{pmatrix}$$

contains the estimation error e_1 of the output z. Note that the sizes of the ellipses $\widetilde{\mathcal{E}}$ and $\mathcal{E}_*$ (in terms of the trace criterion) differ by less than 9%.

In Fig. 2, the solid line shows the nonfragile bounding ellipse $\widetilde{\mathcal{E}}$; the dashed line, the minimal bounding ellipse $\mathcal{E}_*$; the dotted line, the projection of the initial state ellipsoid onto the plane (v_1, v_2), representing an ellipse with the matrix $C_1 P_0 C_1^{\mathrm{T}}$ (see Remark 1).

Now we subject the optimal filter matrix L_* to the perturbation

$$\Delta = \begin{pmatrix} -0.0171 & 0.1641 \\ -0.0714 & -0.4232 \\ -0.9640 & -0.1353 \\ 0.2461 & -0.7643 \end{pmatrix}, \quad \|\Delta\| = 1.$$

In Fig. 3, the dotted line shows the trajectory of the residual e_1 under some initial condition $x_0 \in \mathcal{E}_0$ (the circle indicates its projection onto the plane (v_1, v_2)), some admissible exogenous

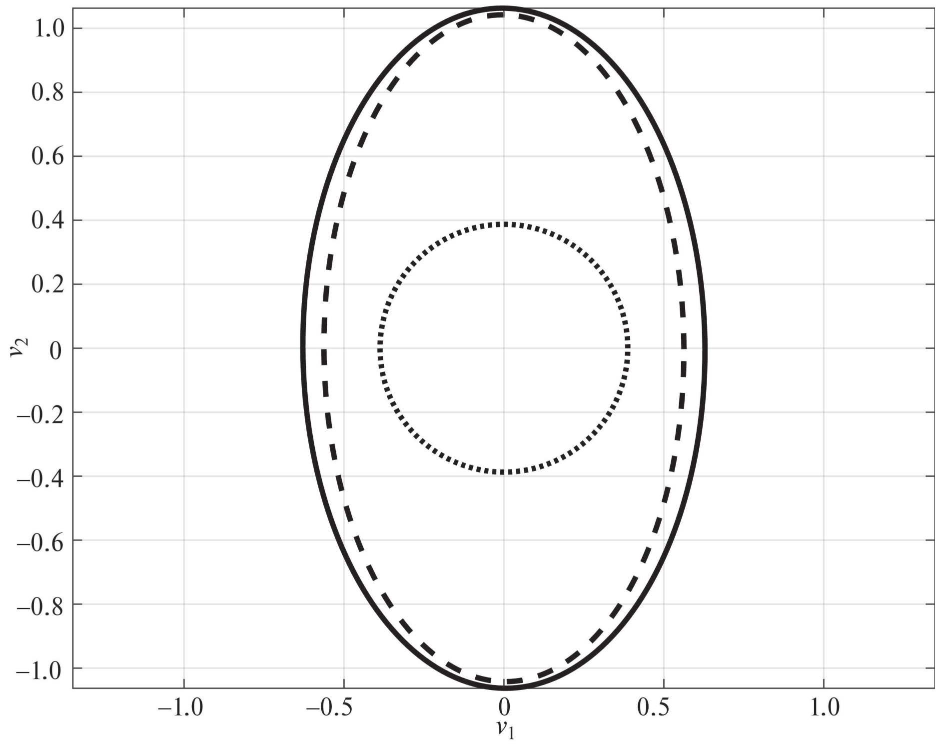

Fig. 2. Bounding ellipses.

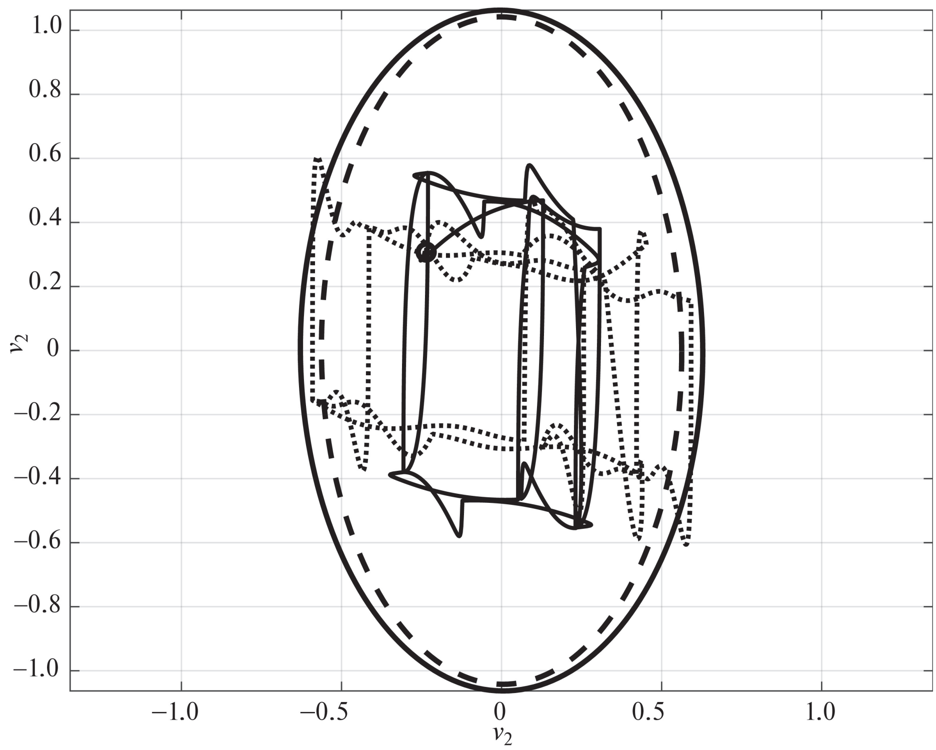

Fig. 3. Residual trajectories.

disturbance w, and the filter matrix $(L_* + \Delta)$. Clearly, the trajectory leaves the minimal bounding ellipse $\mathcal{E}_*$ and, moreover, the nonfragile bounding ellipse $\widetilde{\mathcal{E}}$, despite that the range of uncertainty in the filter matrix is half the nonfragility level $\gamma = 2$.

For comparison, we subject the nonfragile filter matrix $\widetilde{L}$ to a perturbation of twice the magnitude ($\|2\Delta\| = 2 = \gamma$). The solid line in Fig. 3 shows the resulting trajectory of the system residual e_1 under the same initial condition and exogenous disturbance; clearly, the behavior relative to the bounding ellipse is fundamentally different.

All calculations were performed in MATLAB using the `cvx` package [17].

6. CONCLUSIONS

This paper has proposed a simple and universal approach to the nonfragile filtering of arbitrary bounded exogenous disturbances. The approach involves an observer and the method of invariant ellipsoids. With this concept, the original problem has been reformulated in terms of LMIs and reduced to a parametric semidefinite programming problem easily solved numerically. The effectiveness of the filtering method has been demonstrated using an example of a double-spring pendulum.

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APPENDIX

Proof of Theorem 2. For the perturbed filter matrix $(L+\Delta)$, the residual e of system (1) satisfies the differential equation

$$\dot{e} = (A - (L + \Delta)C)e + (D_1 - (L + \Delta)D_2)w. \tag{A.1}$$

According to [6], for the dynamic system (A.1), the invariance condition of an ellipsoid with a matrix $P = Q^{-1} \succ 0$ is equivalent to the existence of a scalar $\alpha(\Delta) > 0$ such that the LMI

$$\begin{pmatrix} \big(A - (L+\Delta)C\big)^{\mathrm{T}} Q + Q\big(A - (L+\Delta)C\big) + \alpha(\Delta)Q & Q\big(D_1 - (L+\Delta)D_2\big) \\ \big(D_1 - (L+\Delta)D_2\big)^{\mathrm{T}} Q & -\alpha(\Delta)I \end{pmatrix} \preccurlyeq 0 \tag{A.2}$$

holds under all admissible matrix uncertainties $\Delta\colon \|\Delta\| \leqslant \gamma$.

Suppose that for some $\alpha > 0$, inequality (A.2) is valid under all admissible Δ; then (A.2) can be written as

$$\begin{pmatrix} (A - LC)^{\mathrm{T}} Q + Q(A - LC) + \alpha Q & Q(D_1 - LD_2) \\ (D_1 - LD_2)^{\mathrm{T}} Q & -\alpha I \end{pmatrix}$$
$$+ \begin{pmatrix} Q \\ 0 \end{pmatrix} \Delta \begin{pmatrix} -C & -D_2 \end{pmatrix} + \begin{pmatrix} -C^{\mathrm{T}} \\ -D_2^{\mathrm{T}} \end{pmatrix} \Delta^{\mathrm{T}} \begin{pmatrix} Q & 0 \end{pmatrix} \preccurlyeq 0.$$

By Petersen's lemma [16] (also, see [6, Corollary 2.2.6]), this inequality holds iff there exists a scalar $\varepsilon > 0$ such that

$$\begin{pmatrix} (A - LC)^{\mathrm{T}} Q + Q(A - LC) + \alpha Q & Q(D_1 - LD_2) \\ (D_1 - LD_2)^{\mathrm{T}} Q & -\alpha I \end{pmatrix}$$
$$+ \varepsilon \begin{pmatrix} -C^{\mathrm{T}} \\ -D_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} -C & -D_2 \end{pmatrix} + \frac{\gamma^2}{\varepsilon} \begin{pmatrix} Q \\ 0 \end{pmatrix} \begin{pmatrix} Q & 0 \end{pmatrix} \preccurlyeq 0.$$

Using Schur's complement lemma, we therefore obtain

$$\begin{pmatrix} (A-LC)^{\mathrm{T}}Q+Q(A-LC)+\alpha Q+\varepsilon C^{\mathrm{T}}C & Q(D_1-LD_2)+\varepsilon C^{\mathrm{T}}D_2 & \gamma Q \\ (D_1-LD_2)^{\mathrm{T}}Q+\varepsilon D_2^{\mathrm{T}}C & -\alpha I+\varepsilon D_2^{\mathrm{T}}D_2 & 0 \\ \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0.$$

With the new matrix variable $Y = QL$, this expression can be reduced to the linear form

$$\begin{pmatrix} A^{\mathrm{T}}Q+QA-YC-C^{\mathrm{T}}Y^{\mathrm{T}}+\alpha Q+\varepsilon C^{\mathrm{T}}C & QD_1-YD_2+\varepsilon C^{\mathrm{T}}D_2 & \gamma Q \\ D_1^{\mathrm{T}}Q-D_2^{\mathrm{T}}Y^{\mathrm{T}}+\varepsilon D_2^{\mathrm{T}}C & -\alpha I+\varepsilon D_2^{\mathrm{T}}D_2 & 0 \\ \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0. \tag{A.3}$$

Since $e_1 = C_1 e$, the residual e_1 is contained in the bounding ellipsoid with the matrix $C_1Q^{-1}C_1^{\mathrm{T}}$. Thus, we arrive at the problem

$$\min \operatorname{tr} C_1Q^{-1}C_1^{\mathrm{T}} \quad \text{subject to the constraints (A.3) and } Q \succ 0.$$

According to [6], this problem is equivalent to minimizing $\operatorname{tr} H$ subject to the constraints (A.3) and

$$\begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0,$$

where $H \in \mathbb{S}^r$ is an auxiliary matrix variable. The proof of Theorem 2 is complete.

Proof of Theorem 4. For the perturbed filter matrix $(L+\Delta)$, the residual of system (4) satisfies the difference equation

$$e_{k+1} = (A-(L+\Delta)C)e_k + (D_1-(L+\Delta)D_2)w_k. \tag{A.4}$$

According to [6], for the dynamic system (A.4), the invariance condition of an ellipsoid with a matrix $P = Q^{-1} \succ 0$ is equivalent to the existence of a scalar $\alpha(\Delta) > 0$ such that the LMI

$$\begin{pmatrix} -\alpha(\Delta)Q & (A-(L+\Delta)C)^{\mathrm{T}}Q & 0 \\ Q(A-(L+\Delta)C) & -Q & Q(D_1-(L+\Delta)D_2) \\ 0 & (D_1-(L+\Delta)D_2)^{\mathrm{T}}Q & -(1-\alpha(\Delta))I \end{pmatrix} \preccurlyeq 0 \tag{A.5}$$

holds under all admissible matrix uncertainties $\Delta\colon \|\Delta\| \leqslant \gamma$.

Suppose that for some $\alpha > 0$, inequality (A.5) is valid under all admissible Δ; then (A.5) can be written as

$$\begin{pmatrix} -\alpha Q & (A-LC)^{\mathrm{T}}Q & 0 \\ Q(A-LC) & -Q & Q(D_1-LD_2) \\ 0 & (D_1-LD_2)^{\mathrm{T}}Q & -(1-\alpha)I \end{pmatrix}$$
$$+\begin{pmatrix} 0 \\ Q \\ 0 \end{pmatrix} \Delta \begin{pmatrix} -C & 0 & -D_2 \end{pmatrix} + \begin{pmatrix} -C^{\mathrm{T}} \\ 0 \\ -D_2^{\mathrm{T}} \end{pmatrix} \Delta^{\mathrm{T}} \begin{pmatrix} 0 & Q & 0 \end{pmatrix} \preccurlyeq 0.$$

By Petersen's lemma, this inequality holds iff there exists a scalar $\varepsilon > 0$ such that

$$\begin{pmatrix} -\alpha Q & (A-LC)^{\mathrm{T}}Q & 0 \\ Q(A-LC) & -Q & Q(D_1-LD_2) \\ 0 & (D_1-LD_2)^{\mathrm{T}}Q & -(1-\alpha)I \end{pmatrix}$$
$$+\varepsilon\begin{pmatrix} -C^{\mathrm{T}} \\ 0 \\ -D_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} -C & 0 & -D_2 \end{pmatrix} + \frac{\gamma^2}{\varepsilon}\begin{pmatrix} 0 \\ Q \\ 0 \end{pmatrix} \begin{pmatrix} 0 & Q & 0 \end{pmatrix} \preccurlyeq 0.$$

Using Schur's complement lemma, we accordingly have

$$\begin{pmatrix} -\alpha Q + \varepsilon C^{\mathrm{T}} C & (A - LC)^{\mathrm{T}} Q & \varepsilon C^{\mathrm{T}} D_2 & 0 \\ Q(A - LC) & -Q & Q(D_1 - LD_2) & \gamma Q \\ \varepsilon D_2^{\mathrm{T}} C & (D_1 - LD_2)^{\mathrm{T}} Q & -(1-\alpha) I + \varepsilon D_2^{\mathrm{T}} D_2 & 0 \\ 0 & \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0.$$

With the new matrix variable $Y = QL$, this expression can be reduced to the linear form

$$\begin{pmatrix} -\alpha Q + \varepsilon C^{\mathrm{T}} C & (QA - YC)^{\mathrm{T}} & \varepsilon C^{\mathrm{T}} D_2 & 0 \\ QA - YC & -Q & QD_1 - YD_2 & \gamma Q \\ \varepsilon D_2^{\mathrm{T}} C & (QD_1 - YD_2)^{\mathrm{T}} & -(1-\alpha) I + \varepsilon D_2^{\mathrm{T}} D_2 & 0 \\ 0 & \gamma Q & 0 & -\varepsilon I \end{pmatrix} \preccurlyeq 0. \tag{A.6}$$

Similar to the continuous-time case, we finally arrive at the optimization problem

$$\min \operatorname{tr} C_1 Q^{-1} C_1^{\mathrm{T}} \quad \text{subject to the constraints (A.6) and } Q \succ 0.$$

This problem is equivalent to minimizing $\operatorname{tr} H$ subject to the constraints (A.6) and

$$\begin{pmatrix} H & C_1 \\ C_1^{\mathrm{T}} & Q \end{pmatrix} \succcurlyeq 0,$$

where $H \in \mathbb{S}^r$ is an auxiliary matrix variable. The proof of Theorem 4 is complete.

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